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Fekete Potentials and Polynomials for Continua

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For planar continua, upper and lower bounds are given for the growth of the associated Fekete potentials, polynomials and energies. The main result is that for continua K of capacity 1 whose outer boundary is an analytic Jordan curve, the family of Fekete polynomials is bounded on K.

Our work makes use of precise results of Pommerenke on the growth of the discriminant and on the distribution of the Fekete points. We also use potential theory, including the exterior Green function with pole at infinity. The Lipschitz character of this function determines the separation of the Fekete points. © 2001 Academic Press

1. INTRODUCTION

Let K be a bounded continuum in the complex plane $\mathbb{C}$. In honor of Fekete [2], an N-tuple of points $(\zeta_1, ..., \zeta_N) \subset K$ (with $N \geqslant 2$) is called a Fekete N-tuple for K if it maximizes the (absolute value of the) discriminant,

$$\Delta_N(K) = \prod_{j,\, k=1,\, j \neq k}^{N} |\zeta_j - \zeta_k| \stackrel{\text{def}}{=} \max_{z_1, ..., z_N \in K} \prod_{j,\, k=1,\, j \neq k}^{N} |z_j - z_k|. \qquad (1)$$

The unbounded component of the complement K^c of K in $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is denoted by E. The Fekete N-tuples for K lie on its outer boundary $\Gamma = \partial E$;

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the continua K and Γ have the same Fekete N-tuples. The union of the bounded components of the complement Γ^c will be called D.

We recall some results of Fekete. The "N-th order diameters" of K,

$$d_N(K) = \Delta_N(K)^{1/N(N-1)}, \tag{2}$$

form a decreasing sequence whose limit $d(K)$ is called the transfinite diameter of K. The number $d(K)$ is equal to the exterior mapping radius $R(K)$ which is defined as follows. Let $w = \Phi(z)$ be the $1-1$ conformal map from E to the exterior of some disc $|w| \leqslant R$, normalized such that $\Phi(z) = z + \mathcal{O}(1)$ around infinity. Then $R = R(K) = d(K)$. Potential theory shows that $d(K) = \operatorname{cap} K$, the logarithmic capacity of K, see the classical paper by Pólya and Szegő [13] and cf. the survey [8]. Precise results on $\Delta_N(K)$ and the distribution of Fekete points have been obtained by Pommerenke [14–17], cf. Sections 4 and 6 below.

For every Fekete N-tuple $Z_N = (\zeta_1, ..., \zeta_N) \subset K$ there is an N-th order "Fekete measure" $\omega_N = \omega_N(Z_N)$, obtained by assigning masses or charges $1/N$ to each of the points $\zeta_1, ..., \zeta_N$. The corresponding "Fekete potential"

$$U^{\omega_N}(z) = \int_K \log \frac{1}{|z-\zeta|}\, d\omega_N(\zeta) = \frac{1}{N} \sum_{k=1}^{N} \log \frac{1}{|z-\zeta_k|} \tag{3}$$

may be compared with the classical equilibrium potential $U^{\omega}(z)$, where ω is the probability measure on K of minimal energy. We summarize what is known and what we can derive from known results with the aid of simple potential theory.

THEOREM 1.1. *For every bounded continuum $K \subset \mathbf{C}$ there are constants c_1 and c_2 such that for all $N \geqslant 2$ and every N-th order Fekete measure ω_N on K,*

$$\log \min\{N^{-3}, d(z, \Gamma)\} - c_1 \leqslant N\{U^{\omega}(z) - U^{\omega_N}(z)\}$$
$$\leqslant \log N + \log\log N + c_2, \qquad \forall z \in \mathbb{C}. \tag{4}$$

If K is convex, or the outer boundary of K is a Jordan curve Γ of bounded rotation or of class $C^{2,\varepsilon}$, the term $\log\log N$ *in the upper bound may be omitted. If Γ is smooth, the quantity N^{-3} in the lower bound may be replaced by N^{-2}.*

The upper bound in (4) for arbitrary continua is equivalent to a result of Pommerenke [15] for polynomials, cf. the upper bound in (7). The sharper upper bound $\log N + c_2$ for special continua may be derived from results of Pommerenke on the discriminant $\Delta_N(K)$ for such continua [14–16], see Section 4 below. The lower bound in (4) depends on the separation of the Fekete points, see Sections 3 and 5. Here we also refer to recent work by Götz and Saff [4] on a more general problem for which one can not expect results as precise as Theorem 1.1. In earlier work the authors obtained estmates for the case of $\mathbf{R}^k$ which for $k=2$ are less precise than (4), cf. [12] and [10].

The inequalities (4) may be restated in terms of the Fekete polynomials

$$F_N(z) = F_N(Z_N, z) = \prod_{k=1}^{N} (z - \zeta_k). \tag{5}$$

It is well-known that the branch of $F_N(z)^{1/N}$ which behaves like z at ∞ converges to the mapping function $\Phi(z)$ on E as $N \to \infty$, cf. Hille [5], Walsh [22]. For a study of the degree of approximation one may compare $|F_N(z)|$ with the N-th power of the function

$$M_\Phi(z) \overset{\text{def}}{=} \begin{cases} |\Phi(z)| & \text{for} \quad z \in E, \\ \operatorname{cap} K & \text{for} \quad z \in \bar{D} = D \cup \Gamma. \end{cases} \tag{6}$$

By (3) and (5) $\log |F_N| = -NU^{\omega_N}$, while $\log M_\Phi = -U^\omega$ (cf. Section 3). Thus with $e^{-c_1} = A$ and $e^{c_2} = B$, (4) may be restated as

$$A \min\{N^{-3}, d(z, \Gamma)\} \leqslant |F_N(z)|/M_\Phi(z)^N \leqslant BN \log N \tag{7}$$

for all N and all $z \in \mathbb{C}$. As indicated before, the general upper bound is due to Pommerenke [15]; his work also shows that the upper bound in (7) can never be less than 1. For the special continua mentioned in Theorem 1.1 the factor $\log N$ in the upper bound may be omitted, see Section 4. It is an interesting question for which continua one may also omit the factor N. By change of scale and the maximum principle, this question is equivalent to the following.

Question 1.2. For which continua K of capacity 1 is the family of Fekete polynomials uniformly bounded on K?

Examples of "good" continua are closed discs and segments. Our main result below will be derived from Pommerenke's precise description [17] of the distribution of the Fekete points for analytic curves (Section 6).

THEOREM 1.3. *The Fekete polynomials form a bounded family on any compact set K of capacity* 1 *whose outer boundary Γ is an analytic Jordan curve.*

It is plausible that less smoothness will suffice, especially in the convex case.

2. RELEVANT POTENTIAL THEORY

Continuing with the notation of Section 1 we restrict ourselves to potential theory for bounded continua $K \subset \mathbf{C}$, although much of the following is valid for arbitrary compact sets of positive capacity, cf. Frostman [3]. For a probability measure μ on K the potential U^μ and the energy $I(\mu)$ are given by

$$U^\mu(z) = \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta), \qquad I(\mu) = \int_{K\times K} \log \frac{1}{|z-\zeta|}\, d\mu(z)\, d\mu(\zeta).$$

The set K carries a unique probability measure ω of minimal energy, the so-called equilibrium distribution:

$$I(\omega) = V = \min I(\mu), \qquad \mu \text{ probability measure on } K.$$

The support of ω coincides with the outer boundary Γ of K.

For the continuum K the exterior domain E (including ∞) is regular for the Dirichlet problem. This fact implies that $U^\omega(z)$ is equal to V everywhere on $\bar{D}$. The constant $V = V(K)$ is called the Robin constant; in terms of V,

$$\operatorname{cap} K = e^{-V}. \tag{1}$$

Discrete measures. Let $\mu_N = \mu_N(z_1, ..., z_N)$ be the probability measure corresponding to the system of N masses or charges $1/N$ at the (distinct) points $z_1, ..., z_N$. The corresponding "discrete energy" is

$$I^*(\mu_N) = \frac{1}{N^2} \sum_{j,\,k=1,\, j\neq k}^{N} \log \frac{1}{|z_j - z_k|}.$$

It follows from Section 1 that the discrete energy $I^*(\mu_N)$ is minimal for N-tuples $(z_1, ..., z_N) \subset K$ if μ_N is an N-th order Fekete measure $\omega_N = \omega_N(\zeta_1, ..., \zeta_N)$. The corresponding discrete energy is equal to

$$I^*(\omega_N) = \frac{1}{N^2} \sum_{j,\,k=1,\, j\neq k}^{N} \log \frac{1}{|\zeta_j - \zeta_k|} = -\frac{1}{N^2} \log \Delta_N(K). \tag{2}$$

As $N \to \infty$, $I^*(\omega_N) \to I(\omega)$ and the Fekete measures ω_N are weak*-convergent to ω, cf. Frostman [3]. In the case where $\Gamma = \partial E$ is a Jordan curve, it follows that

$$\omega_N(\gamma) \to \omega(\gamma) \qquad \text{for every subarc} \quad \gamma \subset \Gamma.$$

Remark. Conformal mapping shows that ω may be interpreted as the *harmonic measure* on Γ relative to E, evaluated at ∞.

3. GREEN FUNCTION AND SEPARATION OF FEKETE POINTS

The exterior domain E of the continuum K has a "Green function" $g(z)$ with pole at infinity: by definition it is harmonic on E except at ∞, where it behaves like $\log|z| + \mathcal{O}(1)$, and it has boundary values 0 on Γ. One has $g(z) = \log\{|\Phi(z)|/\text{cap } K\}$ on E and it is convenient to extend $g(z)$ to a continuous function on $\mathbf{C}$ by the formula

$$g(z) = V - U^{\omega}(z). \tag{1}$$

If Γ is a Jordan curve of class $C^{1,\varepsilon}$ (or just Dini-smooth), classical results on conformal mapping imply that g is of class Lip 1 around Γ, cf. Pommerenke's book [18]. For arbitrary continua one may adapt a result of Johnston [6], Theorem 1, to obtain a suitable Lipschitz result.

PROPOSITION 3.1. *Let Ω be a bounded simply connected domain and let u be harmonic on Ω and continuous on $\bar{\Omega}$. Suppose in addition that u is of class* Lip λ *with $\lambda > \frac{1}{2}$ on the boundary $\partial\Omega$ in a neighborhood of z_0. Then u is of class* Lip $\frac{1}{2}$ *in a neighborhood of z_0 in $\bar{\Omega}$.*

Johnston's Lipschitz hypothesis and conclusion were global in character, but his proof may be adjusted to give the local result.

COROLLARY 3.2. *For an arbitrary bounded continuum K, the Green function g and the equilibrium potential U^{ω} (which are constant on $\bar{D}$) are of class* Lip $\frac{1}{2}$ *on a neighborhood of the outer boundary Γ.*

Proof. For $z_0 \in \Gamma$ one can find a closed disc containing Γ whose boundary C meets Γ somewhere away from z_0. Now apply Proposition 3.1 to g on the (or a) simply connected domain bounded by Γ and C which has z_0 as a boundary point.

The case of a segment shows that the (general and global) result Lip $\frac{1}{2}$ is sharp. The corollary could also be derived from work by Siciak, see [20] Lemma 1.

Separation Result. For general and various special continua the separation of the Fekete points has been discussed by Kővari and Pommerenke [11]. Here we emphasize the connection with the Lipschitz character of g, cf. Götz and Saff [4].

PROPOSITION 3.3. *Suppose g is in* Lip λ *around Γ (locally) with $\frac{1}{2} \leqslant \lambda \leqslant 1$. Then there is a constant $\delta > 0$ such that for all N and every N-th order Fekete N-tuple $(\zeta_1, ..., \zeta_N)$ on K (locally)*

$$\min_{j \neq k} |\zeta_j - \zeta_k| \geqslant \delta / N^{1/\lambda}. \tag{2}$$

Proof. By the maximum property of Fekete points, the polynomial

$$p_k(z) = \prod_{j=1,\ j\neq k}^{N} \frac{z-\zeta_j}{\zeta_k-\zeta_j}$$

of degree $N-1$ has absolute value $\leqslant 1$ on Γ. Now $\log |p_k(z)| - (N-1)\, g(z)$ is harmonic on E and $\leqslant 0$ on Γ, hence $\leqslant 0$ on E. The same bound holds on $\bar{D}$. Thus since $g(\zeta_k)=0$, the Lipschitz property of g implies that for some constant A,

$$\begin{aligned}\log |p_k(z)| &\leqslant (N-1)\{g(z)-g(\zeta_k)\} \leqslant NA\, |z-\zeta_k|^\lambda \\ &\leqslant A \qquad \text{for} \quad |z-\zeta_k| \leqslant N^{-1/\lambda}.\end{aligned}$$

For $|w-\zeta_k| \leqslant \frac{1}{2} N^{-1/\lambda}$ Cauchy's inequality next gives

$$|p_k'(w)| \leqslant \frac{1}{2\pi} \int_{|z-\zeta_k| = N^{-1/\lambda}} \frac{|p_k(z)|}{|z-w|^2}\, |dz| \leqslant 4e^A N^{1/\lambda}.$$

Now let $j\neq k$. If $|\zeta_j-\zeta_k| > \frac{1}{2} N^{-1/\lambda}$ there is nothing to prove. Otherwise $|\zeta_j-\zeta_k| \leqslant \frac{1}{2} N^{-1/\lambda}$ and then a good lower bound is obtained (as in [11]) from the formula

$$1 = |p_k(\zeta_j) - p_k(\zeta_k)| = \left| \int_{\zeta_k}^{\zeta_j} p_k'(w)\, dw \right| \leqslant |\zeta_j-\zeta_k|\; 4e^A N^{1/\lambda}.$$

The above proof confirms the lower bounds of *Kővari and Pommerenke* [11] for the distance between N-th order Fekete points on bounded continua (δ/N^2) and on $C^{1,\varepsilon}$-smooth (or slightly less smooth) Jordan curves (δ/N). For the smooth and the convex case they have also shown that the distance between neighboring points in Fekete N-tuples is bounded above by $const/N$.

4. THE UPPER BOUNDS IN THEOREM 1.1

In the following c, B, c', c'' will denote constants depending only on K; they may vary with the context. For the derivation of the upper bounds we summarize certain results of Pommerenke [14, 16] for the discriminant $\Delta_N(K)$.

THEOREM 4.1. *For any bounded continuum K*

$$N^N \leqslant \Delta_N(K)/(\operatorname{cap} K)^{N(N-1)} < N^N (4e^{-1}\log N + 4)^N. \tag{1}$$

In the convex, bounded rotation or $C^{2,\varepsilon}$ case the final factor may be replaced by an expression of the form cB^N.

We can now prove

PROPOSITION 4.2. *For all $N \geqslant 2$ and every N-th order Fekete measure $\omega_N = \omega_N(\zeta_1, ..., \zeta_N)$ on K,*

$$\begin{aligned} -(1-N^{-1})\, U^{\omega_N}(z) \leqslant -I^*(\omega_N) &= N^{-2} \log \Delta_N(K) \\ &\leqslant -(1-N^{-1})\, V + N^{-1} \log N \\ &\quad + N^{-1} \log\log N + c/N, \qquad \forall z \in K. \end{aligned} \tag{2}$$

For the special classes of continua in Theorems 1.1 *and* 4.1 *the* log log-*term may be omitted.*

Proof. The minimum property of the Fekete energy implies that for each k,

$$W_k(z) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1,\, j \neq k}^{N} \log \frac{1}{|z-\zeta_j|} \geqslant W_k(\zeta_k), \qquad \forall z \in K. \tag{3}$$

Taking the arithmetic mean of these inequalities for $k = 1, ..., N$ one concludes that on K

$$\frac{N-1}{N}\, U^{\omega_N}(z) \geqslant \frac{1}{N^2} \sum_{j,k=1,\, j \neq k}^{N} \log \frac{1}{|\zeta_j - \zeta_k|} = I^*(\omega_N). \tag{4}$$

The proof is completed by application of formulas (1) and (2) of Section 2 and Theorem 4.1.

The desired upper bounds in Theorem 1.1 now follow from

PROPOSITION 4.3. *For all N and every Fekete measure ω_N on K,*

$$U^{\omega}(z) - U^{\omega_N}(z) \leqslant (\log N)/N + (\log\log N)/N + c/N, \qquad \forall z \in \mathbf{C}. \tag{5}$$

For the special classes of continua in Theorem 1.1 *the* log log-*term may be omitted.*

Proof. By Proposition 4.2 one has an inequality (5) on Γ where $U^{\omega} = V$. Now $U^{\omega} - U^{\omega_N}$ is harmonic both inside and outside Γ (including ∞), so that (5) follows from the maximum principle for all $z \in D \cup E$. The proof for the special continua is similar.

Comparison of Energies. Since $I(\omega) = V$, formulas (1) and (2) of Section 2 and Theorem 4.1 imply

THEOREM 4.4. *For every bounded continuum*

$$(\log N)/N + V/N \leqslant I(\omega) - I^*(\omega_N)$$
$$\leqslant (\log N)/N + (\log \log N)/N + c/N. \qquad (6)$$

For the special continua in Theorem 1.1 *the* log log-*term may be omitted.*

On the circle $C(0, R)$ the Fekete N-tuples are given by equally spaced points. Here

$$I(\omega) = -\log R, \qquad I^*(\omega_N) = -(1 - 1/N) \log R - (\log N)/N.$$

Remark. Not distinguishing between discrete energy and "ordinary" energy, Hille [5], Eq. (16.4.13), erroneously assumed that (in our notation) $I^*(\omega_N) > I(\omega)$.

5. THE LOWER BOUND IN THEOREM 1.1

The method below is similar to one used by Götz and Saff [4]. Integration of inequality (3) in Section 4 with respect to ω shows that

$$W_k(\zeta_k) \leqslant \int W_k \, d\omega = (1/N) \sum_{j=1,\, j \neq k}^{N} U^{\omega}(\zeta_j) = (1 - 1/N) \, V. \qquad (1)$$

The gradient of $\log |z - \zeta|^{-1}$ may be represented by the complex number $-1/(z - \zeta)^*$ where the $*$ denotes the complex conjugate. Thus by the definition of W_k,

$$|\text{grad } W_k(z)| \leqslant \frac{1}{N} \sum_{j=1,\, j \neq k}^{N} \frac{1}{|z - \zeta_j|}.$$

We now take $|z - \zeta_k| \leqslant \delta/2N^2$ where δ is a separation constant for Fekete N-tuples on K as in Proposition 3.3 with $\lambda = \frac{1}{2}$. Then $|z - \zeta_j| \geqslant \delta/2N^2$ for all $j \neq k$, hence

$$|\text{grad } W_k(z)| < 2N^2/\delta.$$

It follows that

$$W_k(z) - W_k(\zeta_k) \leqslant 2/N \qquad \text{for} \quad |z - \zeta_k| = \varepsilon \leqslant \delta/N^3.$$

Hence for these same values of z, cf. formula (3) in Section 1,

$$\begin{aligned} U^{\omega_N}(z) &= W_k(z) + (1/N)\log(1/\varepsilon) \\ &\leqslant W_k(\zeta_k) + (1/N)\log(1/\varepsilon) + 2/N \\ &\leqslant V + (1/N)\log(1/\varepsilon) + (2-V)/N. \end{aligned} \tag{2}$$

In the last step we have used (1). Continuing with the same values of z we now use the Lipschitz property of U^ω from Corollary 3.2:

$$V - U^\omega(z) = U^\omega(\zeta_k) - U^\omega(z) \leqslant c\,|z-\zeta_k|^{1/2} \leqslant c(\delta/N^3)^{1/2}. \tag{3}$$

Combining (2) and (3) we obtain the result

$$N\{U^{\omega_N}(z) - U^\omega(z)\} \leqslant \log(1/\varepsilon) + c' \qquad \text{for} \quad |z-\zeta_k| = \varepsilon \leqslant \delta/N^3. \tag{4}$$

Observe that inequality (4) is independent of k, hence it actually holds on the union of the circles $C(\zeta_k, \varepsilon)$. Now the first member of (4) is subharmonic on the domain Ω_ε obtained from $\overline{\mathbf{C}}$ by omitting the closed discs $\Delta(\zeta_k, \varepsilon)$. For $\varepsilon = \varepsilon_0 = \delta/N^3$ one concludes that

$$N\{U^{\omega_N}(z) - U^\omega(z)\} \leqslant \log(N^3/\delta) + c' = \log N^3 + c'' \qquad \text{throughout } \Omega_{\varepsilon_0}. \tag{5}$$

For z in a disc $\Delta(\zeta_k, \varepsilon_0)$ one has $|z-\zeta_k| = \varepsilon \leqslant \varepsilon_0$ and then inequality (4) shows that

$$N\{U^{\omega_N}(z) - U^\omega(z)\} \leqslant \log(1/\varepsilon) + c' \leqslant \log(1/d(z, \Gamma)) + c'. \tag{6}$$

Combination of (5) and (6) establishes the lower bound in Theorem 1.1 with $\alpha = 3$ and $c_1 = \max(c', c'')$.

If the outer boundary of K is a (Dini-) smooth Jordan curve the separation δ/N of the Fekete points allows $\alpha = 2$.

6. RESULTS ON QUESTION 1.2

It is easy to see that closed discs or circles and segments are "good" in the sense of Question 1.2. For the unit circle one has $F_N(z) = z^N - e^{is}$ where s may be any real number, hence $|F_N(z)| \leqslant 2$ for $|z| = 1$.

For the segment $\Gamma = [-1, 1]$ of capacity $\frac{1}{2}$ one may follow Stieltjes and use equilibrium considerations to obtain a differential equation for $y = F_N(x)$:

$$(1-x^2)\,y'' + N(N-1)\,y = 0. \tag{1}$$

Classical considerations now give

$$F_N(x) = c_N(x^2-1)\, P'_{N-1}(x) = 2^N \binom{2N}{N}^{-1} \{P_N(x) - P_{N-2}(x)\},$$

where P_k denotes the Legendre polynomial of degree k, cf. Szegő's book [21]. A standard inequality will then show that $|F_N(x)| \leqslant c/2^N$ on Γ ([21] Section 7.33).

The case of analytic curves. We assume now that K is an analytic Jordan curve Γ (or the closure of the interior of such a Γ) which has been normalized to capacity 1 by a change of scale. Roughly speaking, the Fekete points then correspond to appropriately shifted roots of unity under the conformal map $w = \Phi(z)$. We let $z = \Psi(w)$ denote the inverse map from the exterior of $C(0, 1)$ to E so that $\Psi(w) = w + \mathcal{O}(1)$ around infinity. Our main result here is Theorem 1.3 or equivalently,

THEOREM 6.1. *On an analytic Jordan curve Γ of capacity* 1 *the Fekete polynomials form a bounded family.*

Proof. The proof proceeds in a number of steps.

Step 1. Associated with the conformal map Ψ there is a representation

$$\log \frac{\Psi(w) - \Psi(v)}{w - v} = -\sum_{m, n=1}^{\infty} a_{mn} w^{-m} v^{-n}, \qquad |w|, |v| \geqslant 1, \tag{2}$$

where one has taken the branch of the logarithm which vanishes at $v = w = \infty$; the coefficients a_{mn} come from the Faber polynomials for Γ. For the moment it will be enough to suppose that Γ is of class $C^{2,\varepsilon}$. Then $\Psi(w)$ will be of class $C^{2,\varepsilon}$ for $|w| \geqslant 1$ (we always assume $0 < \varepsilon < 1$). Moreover, the v-derivative of (2),

$$\frac{\Psi'(v)}{\Psi(v) - \Psi(w)} - \frac{1}{v - w} = \frac{1}{2} \frac{\Psi''(w)}{\Psi'(w)} + \mathcal{O}(|v - w|^{\varepsilon}), \tag{3}$$

will be uniformly bounded for $|w| = |v| = 1$, cf. Pommerenke [18], Section 3.3.

Step 2. We need a simple *numerical integration formula* with remainder for C^1 functions $f(v)$ on the unit circle. Using arbitrary equally spaced points

$$v_k = e^{is_k}, \quad s_k = \sigma + 2\pi k/N, \qquad k = 1, ..., N \tag{4}$$

one has the familiar result (cf. for example [7])

$$\left| \int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi} - \frac{1}{N} \sum_{k=1}^{N} f(v_k) \right| \leqslant \sup |f'(e^{it})| \frac{\pi}{N}. \tag{5}$$

This formula may be used to prove a boundedness result for the "Fejér polynomials"

$$G_N(z) = \prod_{k=1}^{N} (z - z_k), \qquad z_k = \Psi(v_k)$$

which vanish at the conformal images of equally spaced points v_k. Assuming that Γ is of class $C^{2,\varepsilon}$ we may apply (5) to the function $f(v)$ given by the real part of (2). For $|w| = 1$ we thus find

$$\sum_{k=1}^{N} \log \frac{|\Psi(w) - \Psi(v_k)|}{|w - v_k|} = N \int_0^{2\pi} \log \frac{|\Psi(w) - \Psi(e^{it})|}{|w - e^{it}|} \frac{dt}{2\pi} + \mathcal{O}(1),$$

where the constant in the $\mathcal{O}$-term depends only on Γ. By (2) the integral is equal to zero. The conclusion is that for $z \in \Gamma$, so that $|w| = |\Phi(z)| = 1$, and for $N \geqslant 2$,

$$\begin{aligned} \log |G_N(z)| &= \sum_{k=1}^{N} \log |\Psi(w) - \Psi(v_k)| \\ &= \sum_{k=1}^{N} \log |w - v_k| + \mathcal{O}(1) \leqslant \mathcal{O}(1), \end{aligned} \tag{6}$$

again with a constant in the $\mathcal{O}$-term that depends only on Γ.

(The boundedness of the family of Fejér polynomials has actually been proved under weaker conditions on Γ, cf. Curtiss [1], Shen and Zhong [19] and Zhong [23].)

Step 3. In what way do the Fekete points deviate from Fejér points? For analytic curves there is a beautiful result by Pommerenke [17].

THEOREM 6.2. *For every analytic Jordan curve Γ there is a real-analytic function ϕ of period 2π and average zero with the following property. For the Fekete N-tuples*

$$\zeta_k = \Psi(e^{i\theta_k}), \qquad k = 1, ..., N, \quad 0 < \theta_1 < \theta_2 < \cdots < \theta_N \leqslant 2\pi$$

on Γ there are shifts σ (depending on the N-tuple) such that

$$\begin{aligned} &\theta_k = s_k + (1/N)\,\phi(s_k) + \tau_k, \quad s_k = \sigma + 2\pi k/N, \qquad k = 1, ..., N, \\ &\tau_k = \mathcal{O}(\sqrt{\log N}/N^2), \qquad \sum |\tau_k| = \mathcal{O}(1/N). \end{aligned} \tag{7}$$

Here the constants in the $\mathcal{O}$-terms depend only on Γ.

It is convenient to introduce the notations

$$h(t) = t + (1/N)\,\phi(t), \qquad w_k = e^{ih(s_k)}. \tag{8}$$

Then on the circle $\{|w| = 1\}$, by (5),

$$\sum_{k=1}^{N} \log \frac{|\Psi(w) - \Psi(w_k)|}{|w - w_k|} = N \int_0^{2\pi} \log \frac{|\Psi(w) - \Psi \circ e^{ih(t)}|}{|w - e^{ih(t)}|} \frac{dt}{2\pi} + \mathcal{O}(1). \tag{9}$$

This time, the integral need not vanish, but it is $\mathcal{O}(1/N)$: the integral becomes zero if we replace dt by $dh(t) = \{1 + (1/N)\,\phi'(t)\}\,dt$. By the boundedness of the function in (3) and the smallness of the remainders in (7) we may next replace the points $w_k = e^{-i\tau_k} e^{i\theta_k}$ in (9) by $e^{i\theta_k}$. Thus the points $\Psi(w_k)$ are replaced by the Fekete points $\Psi(e^{i\theta_k}) = \zeta_k$. It follows that for $z = \Psi(w) \in \Gamma$,

$$|F_N(z)| = \prod |z - \zeta_k| \leqslant C \prod |w - e^{i\theta_k}|, \tag{10}$$

with a constant C that depends only on Γ.

Step 4. It remains to show that the products $\prod_{k=1}^{N} |w - e^{i\theta_k}|$ form a bounded family on the unit circle. We will first verify that the maximum of such a product is attained roughly halfway between consecutive points $e^{i\theta_k}$.

By rotation one may assume that the product

$$\prod |e^{it} - e^{i\theta_k}| = \prod |2 \sin \tfrac{1}{2}(t - \theta_k)|, \qquad t \text{ real}$$

takes on its maximum value at $t = 0$. Then by differentiation

$$\sum \tfrac{1}{2} \cot \tfrac{1}{2}\,\theta_k = 0. \tag{11}$$

To exploit (11) it is convenient to write $\theta_{N-k} - 2\pi = \theta_{-k}$ and to relabel the Fekete points $\zeta_k = \Psi(e^{i\theta_k})$ in such a way that their indices k run from $[\frac{1}{2}N] - N + 1$ to $[\frac{1}{2}N]$. To simplify the notation we take N even, $N = 2p$. The corresponding θ_k's satisfy the inequalities

$$\theta_{-p+1} < \cdots < \theta_0 < 0 < \theta_1 < \cdots < \theta_p.$$

(If $N=2p+1$ one may add a point θ_{-p} which will not cause a problem in the sequel.) By (7) we have $\theta_p-\theta_0=\pi+\mathcal{O}(1/N)$ and $\theta_1-\theta_{-p+1}=\pi+\mathcal{O}(1/N)$, hence our points θ_k and s_k lie on an interval $[-\pi-\mathcal{O}(1/N), \pi+\mathcal{O}(1/N)]$. Now observe that

$$f(x)=\frac{1}{2}\cot\frac{1}{2}x-\frac{1}{x}$$

is a smooth odd function on a neighborhood of $[-\pi, \pi]$ and that the points θ_k differ from equidistant points $s'_k=s_k+\sigma'$ in $[-\pi, \pi]$ by at most $\mathcal{O}(1/N)$. It follows that

$$\sum_{-p+1}^{p}\left(\frac{1}{2}\cot\frac{1}{2}\theta_k-\frac{1}{\theta_k}\right)=\frac{N}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx+\mathcal{O}(1)=\mathcal{O}(1);$$

cf. for example [7]. Hence by (11)

$$\sum_{-p+1}^{p}\frac{1}{\theta_k}=\mathcal{O}(1) \qquad \text{as} \quad N\to\infty. \tag{12}$$

By (7) one has $\theta_1-\theta_0=2\pi/N+\mathcal{O}(\sqrt{\log N}/N^2)$. We show how to derive from (12) and (7) that

$$\theta_1=\frac{\pi}{N}+o\left(\frac{1}{N}\right), \qquad \theta_0=-\frac{\pi}{N}+o\left(\frac{1}{N}\right). \tag{13}$$

Suppose to the contrary that for some sequence $N\to\infty$ one has $N\theta_1\to 2\pi\alpha$ (so that $0\leqslant\alpha\leqslant 1$) with $\alpha\neq\frac{1}{2}$. Then by (7) for $k\geqslant 0$

$$N\theta_{k+1}/2\pi=\alpha+k+\mathcal{O}(k/N)+o(1), \quad N\theta_{-k}/2\pi=\alpha-1-k+\mathcal{O}(k/N)+o(1).$$

Suppose first that $\alpha\neq 0$ or 1. Combining $1/\theta_{k+1}$ with $1/\theta_{-k}$ and summing over $0\leqslant k\leqslant p-1$, one would obtain for the special $N\to\infty$ that

$$\sum_{-p+1}^{p}\frac{1}{\theta_k}=(1-2\alpha)\,c_\alpha N+o(N) \qquad \text{with} \quad c_\alpha>0,$$

in contradiction to (12). To rule out the possibility that $\alpha=0$ or $\alpha=1$ one may carry out the paired summation just over $1\leqslant k\leqslant p-1$. This will show that $1/\theta_0+1/\theta_1=\mathcal{O}(N)$.

Step 5. It is now easy to complete the proof. Continuing with the set-up of step 4 we will compare the maximum $\prod|1-e^{i\theta_k}|$ of $\prod|w-e^{i\theta_k}|$ on the unit circle with $\prod|1-e^{it_k}|$, where we take $t_k=s_k+\theta_0-s_0-\tau_0$. The comparison product is equal to $|1-e^{iN(\theta_0-\tau_0)}|$ which is bounded by 2. Observe that by (7)

$$\theta_k - t_k = \theta_k - s_k - \theta_0 + s_0 + \tau_0$$
$$= \frac{1}{N}\{\phi(s_k) - \phi(s_0)\} + \tau_k = \mathcal{O}(|k|/N^2) + \tau_k. \tag{14}$$

In particular t_k and θ_k will have the same sign for all large N.

The logarithm of the quotient of the products is

$$\sum_{-p+1}^{p} \log \frac{\sin\frac{1}{2}\theta_k}{\sin\frac{1}{2}t_k} \leqslant \sum_k \frac{\sin\frac{1}{2}\theta_k - \sin\frac{1}{2}t_k}{\sin\frac{1}{2}t_k} \leqslant \sum_k \frac{|\theta_k - t_k|}{2\,|\sin\frac{1}{2}t_k|}. \tag{15}$$

In the final sum, the numerators of the terms with $1 \leqslant |k| \leqslant p$ are bounded by const $\cdot\, \{|k|/N^2) + |\tau_k|\}$, cf. (14), while the denominators are bounded from below by $c\,|k|/N$ with positive c. Since the term with $k=0$ remains bounded as $N \to \infty$, it follows that the sums (15) are $\leqslant \mathcal{O}(1)$.

Conclusion. The products $\prod_{k=1}^{N} |w - e^{i\theta_k}|$ are bounded on the circle $C(0, 1)$ by a constant independent of N, and by (10) the same is true for the Fekete polynomials on Γ.

Remark. The same proof works for the "roundish" curves of class $C^{4,\varepsilon}$ considered in [7]. Here the θ_k's of the Fekete points $\Psi(e^{i\theta_k})$ satisfy (7) with $\tau_k = \mathcal{O}(1/N^2)$. The proof can probably be made to work for all $C^{4,\varepsilon}$ curves by further refinement of Pommerenke's method ([17, part II]) as used in [9].

7. FURTHER DISCUSSION ON QUESTION 1.2

As before, let Γ be the outer boundary of K and cap $\Gamma = 1$. By the maximum property of Fekete points,

$$M_k(z) \stackrel{\text{def}}{=} \prod_{j=1,\, j\neq k}^{N} |z - \zeta_j| \leqslant M_k(\zeta_k) = |F'_N(\zeta_k)|, \qquad \forall z \in \Gamma \text{ and } \forall k, \tag{1}$$

$$\prod_{k=1}^{N} |F'_N(\zeta_k)| = \Delta_N = \Delta_N(\Gamma). \tag{2}$$

We suppose now that K is convex or smooth. Then by Theorem 4.1,

$$|F'_N(\zeta_h)| \stackrel{\text{def}}{=} \min_k |F'_N(\zeta_k)| \leqslant \Delta_N^{1/N} \leqslant cN. \tag{3}$$

Recall that neighboring Fekete points have distance $\leqslant c'/N$ (Section 3). Hence for $z \in \Gamma$ between ζ_h and a neighbor, cf. (1),

$$|F_N(z)| = |z - \zeta_h| \; M_h(z) \leqslant cc'. \tag{4}$$

Some Speculation. Precise analysis of the Fekete polynomials in the exterior of an analytic curve Γ shows, cf. Pommerenke [17],

$$F_N \approx \Phi^N (\Phi')^{-1/2}, \qquad F'_N \approx N\Phi^{N-1}(\Phi')^{1/2} - (1/2)\, \Phi^N (\Phi')^{-3/2}\, \Phi''.$$

Thus on the level curve $\{|\Phi| = R\}$ in E and for fixed large N,

$$|F_N|/R^N \approx |\Phi'|^{-1/2}, \qquad |F'_N|/NR^{N-1} \approx |\Phi'|^{1/2}. \tag{5}$$

Even if Γ is just (locally) smooth, this should give an indication of the (local) qualitative behavior of $|F_N|$ and $|F'_N|$ on the curve.

Conjecture 7.1. In the convex or smooth case, $|F_N|$ will be maximal on Γ near the Fekete points where $|\Phi'|$ is minimal, hence where the magnification under the map Φ is least. Similarly the relative maxima of $|F'_N|$ will be smallest where $|\Phi'|$ is minimal.

By (4), curves for which this is correct are "good" in the sense of Question 1.2.

Example. The interval $I = [-1, 1]$ revisited. The differential equation (1) in Section 6 for the Fekete polynomial $y = F_N$ may be used to show that $|y|$ is *decreasing* along the nonnegative zeros of y'. Indeed, if one multiplies the differential equation by $2y'$ and then integrates from the zero $\eta_j \geqslant 0$ of y' to the zero $\eta_k > \eta_j$, one finds that

$$N(N-1)\{\, y(\eta_k)^2 - y(\eta_j)^2\} = \int_{\eta_j}^{\eta_k} (x^2 - 1)\; dy'(x)^2 = -\int_{\eta_j}^{\eta_k} 2xy'(x)^2\; dx < 0.$$

Similarly, $|y'|$ is *increasing* along the nonnegative zeros ξ_j of y:

$$\frac{y'(\xi_k)^2 - y'(\xi_j)^2}{N(N-1)} = -\int_{\xi_j}^{\xi_k} \frac{1}{1-x^2}\; dy(x)^2 = \int_{\xi_j}^{\xi_k} \frac{2x}{(1-x^2)^2}\; y(x)^2\; dx > 0.$$

It follows that $|F_N|$ assumes its maximum on I at the first nonnegative zero of F'_N and that the smallest relative maximum of $|F'_N|$ on I occurs at the first nonnegative Fekete point. Since by Theorem 4.1 one has $\Delta_N(I)^{1/N}\, 2^N \leqslant c'$, the argument which led to (4) will confirm that $\max_I |F_N| \leqslant const/2^N$.

For a Jordan curve Γ in the shape of a square one may speculate that the maximum of $|F_N|$ on Γ occurs near the middle of the sides, while the relative maxima of $|F'_N|$ increase from the middle of the sides towards the vertices. If this speculation is correct, a square is also "good".

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